

GUJARAT TECHNOLOGICAL UNIVERSITY**B.E. Sem-II [All Branch] examination June 2009****Subject code: 110009****Subject Name: Maths - II****Date: 17/06/2009****Time: 10:30am-1:30pm****Total Marks: 70****Instructions:**

1. Attempt all questions.
2. Make suitable assumptions wherever necessary.
3. Figures to the right indicate full marks.

Q.1 ATTEMPT THE FOLLOWING:

- (a) Define rank of the matrix. Find the rank of the matrix 03

$$A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix}$$

- (b) Let \mathbb{R}^3 have the Euclidean inner product. Use the Gram-Schmidt process to transform the basis vectors $u_1 = (1, 1, 1)$, $u_2 = (-1, 1, 0)$ and $u_3 = (1, 2, 1)$ into an orthonormal basis $\{v_1, v_2, v_3\}$. 04

- (c) Find the eigenvalues and bases for the eigenspaces of A^{25} and $A+2I$, where 03

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

- (d) i. Show that the functions $f(x) = x$ and $g(x) = \sin x$ form a linearly independent set of vectors in $C^1(-\infty, \infty)$. 01

- ii. Show that if $0 < \theta < \pi$, then $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ has no real

eigenvalues and consequently no eigenvectors.

- iii. Define singular matrix. Find the inverse of the matrix A if it is invertible 02

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Q.2

- (a) Determine the dimension and basis for the solution space of the system $3x_1 + x_2 + x_3 + x_4 = 0$, $5x_1 - x_2 + x_3 - x_4 = 0$ 03

- (b) Justify your answer. Why the following sets are not vector space under the given operations? 02

- i. The set of all pairs of real numbers (x, y) with the operation

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \text{ and } \alpha(x, y) = (2\alpha x, 2\alpha y).$$

- ii. In \mathbb{R}^3 , the operations defined as under

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (z_1 + z_2, y_1 + y_2, x_1 + x_2)$$

- (c) Let λ be an eigenvalue of a matrix A . 02

Then prove that (i) $\lambda + k$ is an eigenvalue of $A + kI$ (ii) $k\lambda$ is an eigenvalue of kA .

- (d) i. Solve the following system by Gauss-Elimination method 04

$$2x + 2y + 2z = 0, -2x + 5y + 2z = 1, 8x + y + 4z = -1$$
- ii. By using Gauss-Jordan elimination, Find the inverse of the given matrix 03
- $$\mathbf{A} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{-2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & \frac{-4}{5} & \frac{1}{10} \end{bmatrix}$$

OR

- (d) I For which values of K and λ the following system have (i) no solution
(ii) unique solution (iii) an infinite no. of solutions 04
- $$x + y + z = 6, x + 2y + 3z = 10, x + 2y + Kz = \lambda$$

II Find basis for the row and column spaces of $A = \begin{bmatrix} 1 & 4 & 5 & 4 \\ 2 & 9 & 8 & 2 \\ 2 & 9 & 9 & 7 \\ -1 & -4 & -5 & -4 \end{bmatrix}$.

Q.3

- (a) Prove that $(M_n(R), +, \bullet)$ is a vector space over \mathbf{R} . 05
- (b) Determine whether the following spans the vector space \mathbf{R}^3 ; 03
- (i) $v_1 = (2, -1, 3), v_2 = (4, 1, 2)$ and $v_3 = (8, -1, 8)$,
(ii) $v_1 = (2, 2, 2), v_2 = (0, 0, 3)$ and $v_3 = (0, 1, 1)$.
- (c) Let M_{22} have the inner product $\langle A, B \rangle = \text{tr}(A^T B)$. Find the cosine of the angle between A and B , where $A = \begin{bmatrix} 2 & 6 \\ 1 & -3 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$ 02
- (d) Show that $V = \{(x, y) / x = 3y\}$ is a subspace of \mathbf{R}^2 . State all possible subspaces of \mathbf{R}^2 . 02
- (e) Find the rank and nullity of the matrix $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ 02

OR

- Q.3** (a) Let $V = \{(x, y) / x, y \in \mathbf{R}, y > 0\}$. Let $(a, b), (c, d) \in V$ and $\alpha \in \mathbf{R}$. Define $(a, b) + (c, d) = (a + c, b \cdot d)$ and $\alpha \cdot (a, b) = (\alpha a, b^\alpha)$. 05
- (b) Define basis of a vector space. Let $v_1 = (1, 0, 0), v_2 = (2, 2, 0)$ and $v_3 = (3, 3, 3)$. Show that the set $S = \{v_1, v_2, v_3\}$ is a basis for \mathbf{R}^3 . 03
- (c) Define inner product space. Let $u = \langle u_1, u_2 \rangle, v = \langle v_1, v_2 \rangle \in \mathbf{R}^2$. Define $\langle u, v \rangle = 4u_1v_1 + u_2v_1 + 4u_1v_2 + 4u_2v_2$. Prove that $(\mathbf{R}^2, \langle \cdot, \cdot \rangle)$ is an inner product space. 05
- (d) Let V be a vector space. For a nonempty set A , prove that $A \subset \text{span}(A)$. 01
- Q.4** (a) Check whether the following transformations are linear or not? 02
- (i) $T : V \rightarrow \mathbf{R}$, where V is an inner product space, and $T(u) = \|u\|$.
(ii) $T : M_{mn} \rightarrow M_{nm}$, where $T(A) = A^T$.

03

- (b) Let $T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$, defined by $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - 2x_2 \\ -x_2 \end{pmatrix}$ and let $B = \{e_1, e_2\}$ and $B' = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \end{pmatrix} \right\}$. Then using $[T]_{B'} = P^{-1}[T]_B P$, find $[T]_{B'}$, where P is the transition matrix from B' to B . 03
- (c) Consider the basis $S = \{v_1, v_2\}$ of \mathbf{R}^2 , where $v_1 = (-2, 1)$ and $v_2 = (1, 3)$ and let $T : \mathbf{R}^2 \rightarrow \mathbf{R}^3$ be the linear transformation such that $T(v_1) = (-1, 2, 0)$ and $T(v_2) = (0, -3, 5)$. Find a formula for $T(x_1, x_2)$, and use that to find $T(2, -3)$. 03
- (d) Define kernel of T . Let $T : \mathbf{R}^3 \rightarrow P_1$ be a linear transformation defined by $T(a_1, a_2, a_3) = (-a_1 + 2a_2 + a_3) + (-a_2 + a_3)x$. Find which of the following vectors are in $\ker(T)$; (i) $u = (6, 2, 2)$, (ii) $u = (2, -1, 1)$ and (iii) $u = (0, 0, 0)$. 02
- (e) If u and v are orthogonal unit vectors, then what is the distance between u and v ? Justify your answer. 02
- (f) Define: Algebraic multiplicity of an eigenvalue. Determine the algebraic and geometric multiplicity of $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ if the eigenvalues of A are $\lambda = 2, -1, -1$ and corresponding eigenvectors for $\lambda = 2$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and for $\lambda = -1$ are $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

OR

- Q.4** (a) State Cauchy-Schwarz inequality. Verify Cauchy-Schwarz inequality for the vectors $u = (-3, 1, 0), v = (2, -1, 3)$. 02
- (b) Prove that $\langle u, v \rangle = \frac{1}{4} \|u + v\|^2 - \frac{1}{4} \|u - v\|^2$ 01
- (c) Let $T : \mathbf{R}^4 \rightarrow \mathbf{R}^3$ be the linear transformation given by the formula $T(x_1, x_2, x_3, x_4) = (4x_1 + x_2 - 2x_3 - 3x_4, 2x_1 + x_2 + x_3 - 4x_4, 6x_1 - 9x_3 + 9x_4)$. Find a basis and dimension of $\ker(T)$, $\text{rank}(T)$ and verify the dimension theorem. 05
- (d) Find the projection of $u = (1, -2, 3)$ along $v = (1, 2, 1)$ in \mathbf{R}^3 . 01
- (e) Let $S, T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be the linear transformations given by the formulas $T(x, y) = (x + y, x - y)$ and $S(x, y) = (2x + y, x - 2y)$. (i) Show that S and T are one to one, (ii) Find formula for $T^{-1}(x, y)$, $S^{-1}(x, y)$ and $(S \circ T)^{-1}(x, y)$, (iii) Verify that $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$. 05
- Q.5** (a) Find a matrix P that diagonalizes $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$, and determine $P^{-1}AP$. 06

(b)

By using Cayley-Hamilton theorem, if $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$, then prove that

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

- (c) Find a change of variables that reduces the quadratic form $2x_1^2 + 2x_2^2 - 2x_1x_2$ to a sum of squares and express the quadratic form in terms of the new variables.

OR**Q.5 (a)**

06

Find an orthogonal matrix P that diagonalizes $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$.

- (b) Find the least squares solution of the linear system $Ax = b$ given by $x_1 + x_2 = 7, -x_1 + x_2 = 0, -x_1 + 2x_2 = -7$ and find the orthogonal projection of b on the column space of A .
- (c) Describe the conic whose equation is $5x^2 - 4xy + 8y^2 - 36 = 0$.

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